Fourier Transform Basics

What is the Fourier Transform?

The Fourier Transform is the measurement of a signal along a set of complete , unit , orthogonal set of basis vectors. Completeness means that all real world signals can be expressed without loss of information in terms of the Fourier basis. Orthogonality implies that any component along one basis gives a zero projection on all other basis,i.e changes in component along one basis doesnot affect the measurement along the other basis.It also turns out that this set of basis is symmetric.

So what is these wonderful set of basis , and why we do we like them so much?

This set is the set of complex spirals, each rotating at multiples of a certain base frequency.This is very useful because complex sinusoids e^{jw} are eigen vectors of all linear systems.

And, since we can represent all real world signals as weighted sum of complex sinusoids, it is possible to represent any real world signal in terms of the Fourier basis.

Complex sinusoids (or spiarls) have certain useful prpoerties.

Most important property is that when a complex sinusoid is passed through any linear time invariant system, it only gets scaled at the output. It remains a complex sinusoid of the same frequency.This is so because all that an LTI system can do is scale and shift the input. A shift of a complex spiral by $'x'$ is same as its multiplication by e^{jx}.('Cork Screw effect') .So , at the outputs this only amounts to a scale of the input.

So , all LTI systems can be expressed in terms of how they scale the complex spirals. This is best known to us as $H(w)$ or the frequency response of the system.

$Note$ :: It is important to realize that a real sinusoid is not an eigen vector of LTI systems, as it gets scaled as well as undergo phase change.

Least Squares approximation in the complex domain

We have already seen, in the 2nd lecture, how we can obtain a real number by which if we scale a real vector U it fits another vector X best in the L2 sense.Now, we will extend it complex numbers, i.e. we will find a complex number A , such that if we scale U by A it best fits X in the L2 sense.

L2 norm best fit of X to Y , means that the root sum of squares(R.S.S) of the difference of Y and X is minimum.

If we look at the vector U as a measurement basis that measures a component of X along its direction, then the complex scale A that we find such that U fits X best in the L2 sense, then A is the Fourier component or the eigen value for eigen vector U in X.

Why is the L2 norm used usually ?

Refer to fig-1.

The y axis is the minimaztion function(remanant vector) and x-axis is the independent variable(scale).

Now we notice that in case of the L1 norm,the difference signal is not differentiable at its minimum point. Such is not the case for the L2 norm, which is parabolic and differentiable at the minima.

Problem ::

Given complex vectors X and U we need to find a complex scalar A such that

$||X - aU||_2^2$

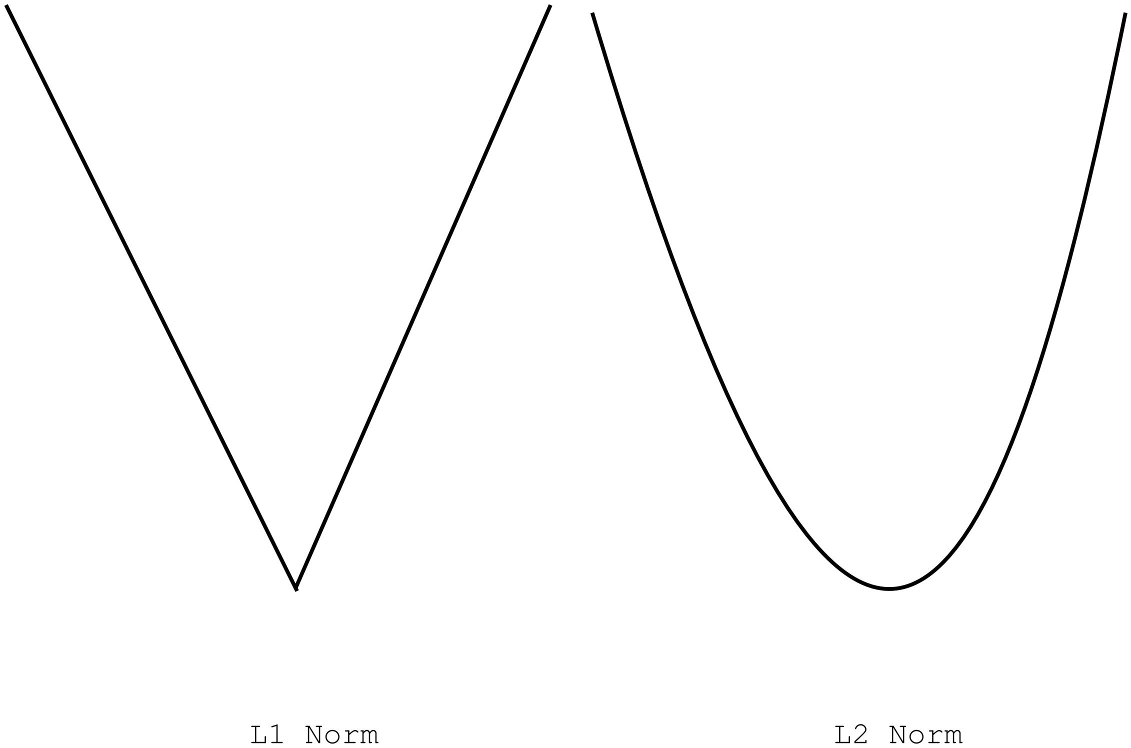

Figure 1: Comparisons of the L1 and L2 norms

is minimum.

Solution ::

$E = \int (X - AU)(\overline{X - AU})$

$E = \int (X - AU)(\overline{X} - \overline{AU})$

$E = \int (X - AU)(\overline{X} - \overline{A}.\overline{U})$

This is because

1. $\overline{(p+q)} = \overline{p} + \overline{q}$

2. $\overline{(p.q)} = \overline{p}.\overline{q}$

$E = \int (X\overline{X} - AU\overline{X} - X\overline{AX} - AU\overline{AU})$

$E = \int X\overline{X} - \int AU\overline{X} - \int A\overline{A}.\overline{U} + \int U\overline{U}$

Rewriting the above equation in a simpler form,

$E = C + BA + \overline{B}.\overline{A} + Aa\overline{A}$

$$E = \begin{pmatrix} A_{real} & A_{img} \end{pmatrix} \begin{pmatrix} 2A & 0 \\ 0 & 2A \end{pmatrix} \begin{pmatrix} A_{real} \\ A_{img} \end{pmatrix} + \begin{pmatrix} 2B & -2B \end{pmatrix} \begin{pmatrix} a_{real} \\ a_{img} \end{pmatrix} + \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}$$

This a quadratic equation in matrices, just like the one we obtained in 2nd lecture. Solving this ,

$Min = (A + \overline{A})^{-1}(2B_{real} - 2B_{img})^T$

$$Min = \begin{pmatrix} 2A_{real} & 0 \\ 0 & A_{img} \end{pmatrix}^{-1} \begin{pmatrix} 2B_{real} \\ 2B_{img} \end{pmatrix}$$

$MinScale = \overline{B}/A$

$MinScale = \int U\overline{X}dt / \int U\overline{U}dt$

$MinScale = < u, x > / < u, u >$

Observation ::

DOT(X , U) is not commutative. In fact it is conjugate of DOT(U , X)

Domain of the 4 types of the F.T

There 4 types of Fourier transforms depending on their domains. Here is the classification.

TRANSFORM :: DOMAIN

Fourier Series :: Continous Finite

Fourier Transform :: Continous Infinite

Discrete F.T :: Discrete Infinite

Discrete Time F.T :: Discrete Finite

Discrete Fourier Transform

Time domain is Discrete and Finite Frequency domain is Discrete and Finite

The Fourier basis here is a set of N discrete time complex spirals rotating at frequency that is integral multiples of 2pi/N.

Equation of the k'th spiral : $u_k[n] = e^{jk2pi/N}$

NOTE :: Different normalization factors are used for these basis vectors, but they only affect the scaling of the components in the other domain. The normalization factor of

$$1/\sqrt{N}$$

, gives same energy in both domains.

For measurement over N such spirals rotating at speeds k = 0 , 1 ,2 ... N-1,the above result the Fourier coefficient matrix X is obatained from x as

$$X = \begin{pmatrix} \overline{u_0} \\ \overline{u_1} \\ \overline{u_2} \\ \overline{u_3} \\ \overline{u_4} \\ \overline{u_5} \\ \vdots \end{pmatrix} x$$

The reverse transformation is ,

$$x = (\, u_0 \quad u_1 \quad . \quad . \quad .\,)\, X$$

We can show that the set of basis are mutually orthogonal.

Lets take two unit complex spirals one that completes w1 turns in N samples, other that takes w2 turns. If we dot product of the two, the resultant is a complex spiral of w1 + w2 frequency. So, over N samples, the spiral completes w1 + w2 rotations. The sum over w1 + w2 cycles is always zero.

To see why the sum of a complex spiral, like the one above, over w1 + w2 cycles is zero ,see the figure below. Assume that N = 8

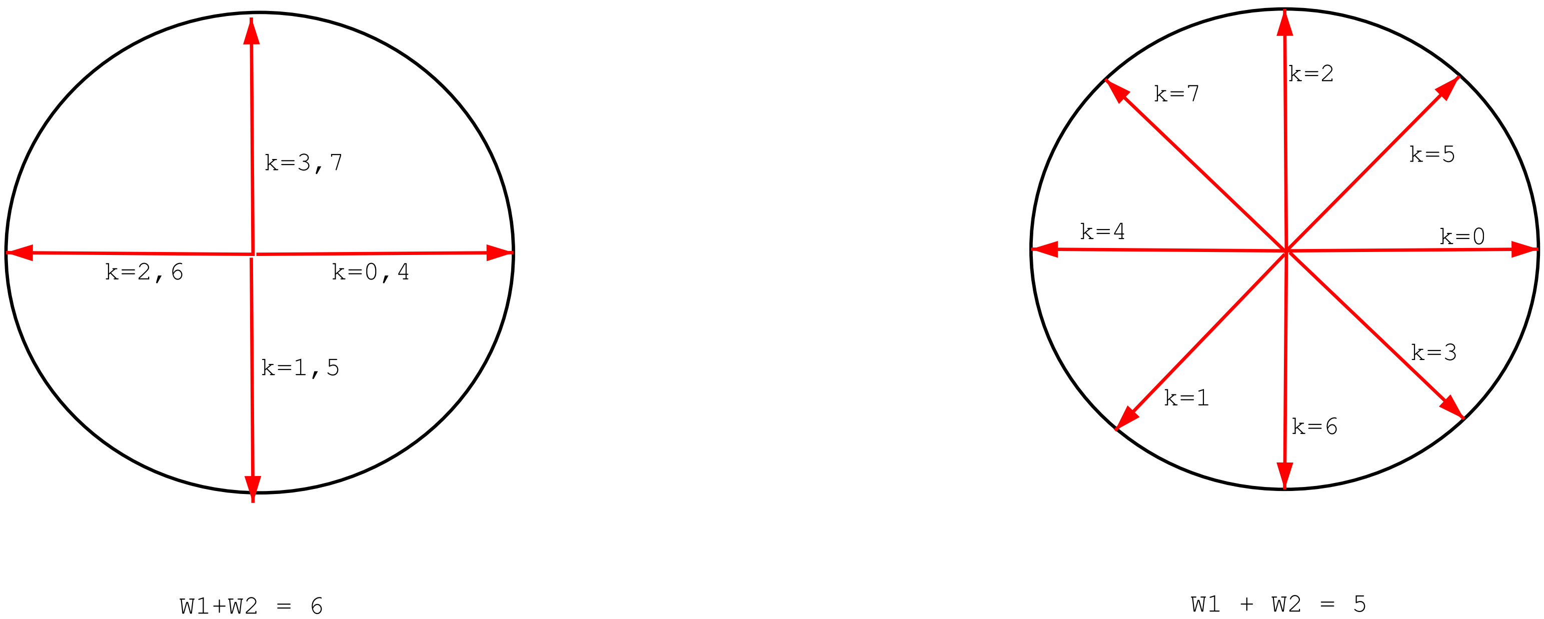

Figure 2: Sum of N=8 spiral over w1+w2 cycles is 0

What we notice as that the vectors are all having their tips along the unit circle,i.e. their magnitudes are the same, and there are pairs of such vectors pointing in excatly opposie directions, their total sum due to cancellation will be 0. Remember that when we add to vectors the addition is by triangle law.So two opposite vectors yeild a zero. The thing to notice is that only such opposite pair of vectors occur.

Of course, the more rigorous algebraic proof is very simple to do. It is just a summation with sqiggly math charcatars, and not as colorful as the one above.

Hence, what is seen is an overview of what Fourier Transform is. Its application and properties will be seen in the next lecture.