Random Processes

30th March 2004

Summary of the lecture

We defined the term *random process*.

We said that we want to predict it, using something called a Wiener filter.

We defined a class of random processes, called stationary random processes, that are especially suited to Wiener prediction.

We defined filtering of a random process, and discussed what happens to a random process when it is filtered.

We defined the power spectrum of a stationary random process.

Random Processes

A random process is a sequence of random variables.

Usually we look at this sequence as time-ordered. Suppose there is this guy with nothing else to do, tossing a coin every second. Associated with each time t, there is a random variable—the result of the coin toss. Before he starts tossing coins all the results are unknown and therefore random variables. At time t_n, the n^{th} random variable is "opened".

Because there are infinitely many random variables in a random process, the PDF is not defined as an infinite dimensional table, but by defining the PDF of every finite subset of the sequence. So there are (infinitely many) finite dimensional tables now.

Prediction of Random Processes and Stationarity

We are interested in random processes because we can use them to model things like speech, and then use this model for compression.

The idea of compression is to predict the next outcome of a random process, and then transmit only the error between the actual outcome and the prediction. If our prediction is good, the error will have much less energy, so fewer bits have to be transmitted.

One method of prediction, called *Wiener prediction*, is to predict an outcome by a linear combination of the previous n outcomes. The coefficients in this linear combination must be constant numbers—we can't change them for every prediction.

Consider what Wiener prediction does:

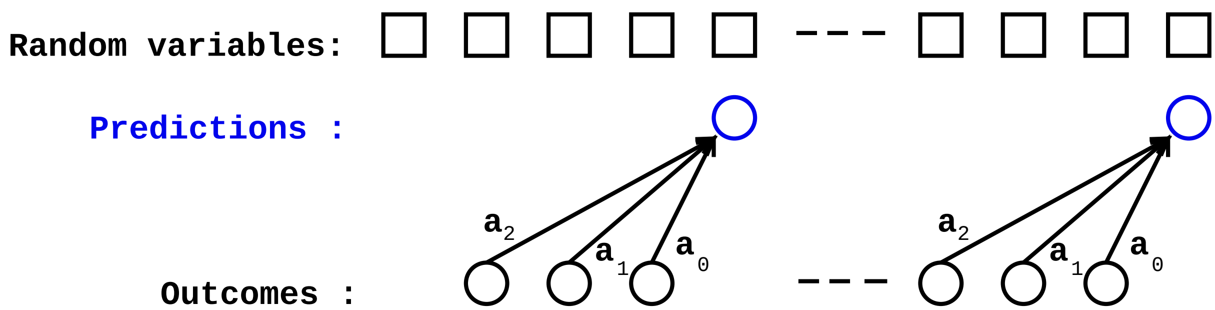

Figure 1: Wiener Prediction

Shown above are two predictions. Each uses the previous three outcomes, weighted by the **same** numbers a_0, a_1, and a_2.

What kind of processes are suitable for this kind of prediction ?

The a's depend on the joint PDF of the random variable being predicted and the one being used for the prediction. So the suitable processes are ones where any random variable X_n has the same joint PDF with the previous three random variables X_{n-1}, X_{n-2}, and X_{n-3}.

Such a process is called *stationary of order 3*. In general you can have a stationary process of order L, where the joint PDF of X_n and X_{n-i} is same for all n for each $i <= L$.

If a process is stationary of order L it is stationary of all orders less than L. A process that is stationary of all orders is called a *strict-sense stationary process*.

Strict-sense stationarity is a very restrictive. Not many real life random processes are strict-sense stationary. Instead, a wider restriction can be made on the random process. Instead of saying that the entire joint PDF of X_n and X_{n-i} should be same for all n, we will merely say that $E[X_n X_{n-i}]$ should be same for all n.

Such a process, where $E[X_n X_{n-i}]$ depends only i, is called a *wide-sense stationary process*.

Henceforth when we say "stationary process", we will mean "wide-sense stationary process".

The auto-correlation function

Define

$$\gamma_{xx}(i) = E[X_0 X_i]$$

Properties worth noting:

1. $\gamma_{xx}(i) = \gamma_{xx}(-i)$. (Because $E[X_0 X_i] = E[X_i X_0]$.)

2. $\gamma_{xx}(0)$ is the variance of X_0 (or any X_i, because they all have equal variance).

3. $\gamma_{xx}(0) \geq \gamma_{xx}(i), \forall i$. (Because nothing can be more correlated with a random variable than that random variable itself.)

4. If $\gamma_{xx}(0) = \gamma_{xx}(k)$, then γ_{xx} is periodic with period with k.

Filtering of Random Processes

Take a random process x, convolve it with a filter f, and you get another random process y. Going to the analogy of closed boxes, this new random process consists of boxes which when opened cause some boxes in the original process to be opened and added up using the gather kernel.

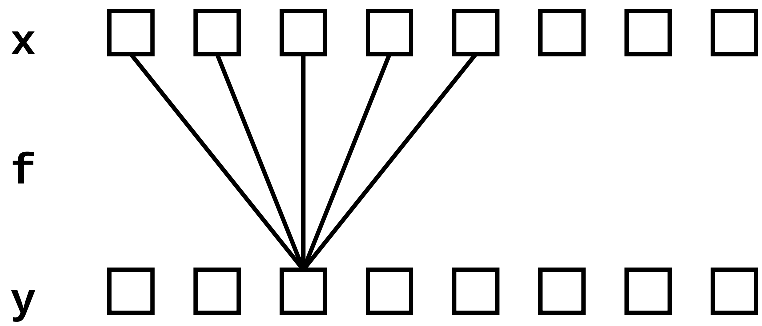

Figure 2: Filtering a random process makes another random process

If x is strict-sense stationary, then so is y.

Define a *white noise process* as one in which all the variables are uncorrelated. The autocorrelation function looks like the Kronecker delta:

$$\gamma_{xx}(i) = \delta_0(i)$$

If x is white noise, what is kind of process is y ?

For example if f is of order 3:

$$Y_0 = f_0 X_{-1} + f_1 X_0 + f_2 X_1$$

Since the $E[X_i X_j] = 0$ for $i \neq j$,

$$E[Y_0^2] = E[f_0^2 X_{-1}^2 + f_1^2 X_0^2 + f_2^2 X_1^2]$$

Therefore:

$$\gamma_{yy}(0) = \sum_i f_i^2$$

Next,

$$E[Y_0 Y_1] = E[(f_0 X_{-1} + f_1 X_0 + f_2 X_1)(f_0 X_0 + f_1 X_1 + f_2 X_2)] = f_0 f_1 + f_1 f_2$$

Therefore:

$$\gamma_{yy}(1) = \sum_i f_i f_{i+1}$$

Similarly,

$$\gamma_{yy}(j) = \sum_i f_i f_{i+j} = r_{ff}(j)$$

And so y is wide-sense stationary.

Now consider what happens if x is wide-sense stationary. We will prove that y is wide-sense stationary and find a relation between γ_{yy}, γ_{xx}, and r_{ff}.

Consider an example f of order 3.

Then:

$$E[Y_0^2] = E[(f_0 X_{-1} + f_1 X_0 + f_2 X_1)(f_0 X_{-1} + f_1 X_0 + f_2 X_1)]$$

$$\gamma_{yy}(0) = E[Y_0^2] = \left(\gamma_{xx}(0) \sum_i f_i^2\right) + \left(\gamma_{xx}(1) \sum_i f_i f_{i+1}\right) + \left(\gamma_{xx}(-1) \sum_i f_i f_{i+1}\right) + ...$$

Next,

$$E[Y_0 Y_1] = E[(f_0 X_{-1} + f_1 X_0 + f_2 X_1)(f_0 X_0 + f_1 X_1 + f_2 X_2)]$$

$$\gamma_{yy}(1) = E[Y_0 Y_1] = \left(\gamma_{xx}(1) \sum_i f_i^2\right) + \left(\gamma_{xx}(0) \sum_i f_i f_{i+1}\right) + ...$$

We can see that :

$$\boxed{\gamma_{yy} = \gamma_{xx} * r_{ff}}$$

In the Fourier domain this is:

$$\Gamma_{yy} = \Gamma_{xx} \circ R_{ff} \qquad \text{where } \circ \text{ denotes pointwise multiplication.}$$

But since $r_{ff} = f \star \overleftarrow{f}$, so $R_{ff} = F \circ \widehat{F} = |F|^2$. So,

$$\Gamma_{yy} = \Gamma_{xx} \circ |F|^2$$

Γ_{xx} is called the *power spectrum* of x.