Lecture 14: Stationarity and the Innovations Process
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Why is noise 'white'?

Most noise is from 'seemingly' independent processes.
eg. The 'hisss' sound caused due to turbulent flow of air seems to be made by independent particles though they are interacting with each other. The same can be said about the weather.
So basically two samples may be chaotically related with each other, giving us the impression that they are actually doing something 'new' every time (i.e. random).

Another important source of noise is *Electron noise*.
There are 2 basic types of e- noise:

1. Ensemble noise: the buzz of the e-
2. Shock Noise[1]: which is a random point process

Why does some noise get colored?

Basically, noise gets coloured due to filtering (as we saw in the previous lecture). So what sort of filters are we talking about here?
Let us look at a *signal passing through the conductor...*

- We have the white noise due to e-
- Then you have the distributive capacitance which acts as a high/ low pass filter depending on current/voltage.
- And of course, you have the induction wall interference[2] (multipath conduction) which causes symbol distortion (overtaking of symbols). We saw this with respect to fibre optic cables, where different wavelengths used to take different paths and suffer different reflections on the walls of the cable.

So these filters acting on the white noise produce enough correlation to give the noise 'colour'.

Innovations!

What we will show now is that given a stationary random process Y having correlation γ_{YY}, we can get a causal, stable filter f such $\gamma_{YY} = r_{ff}$.
What this implies is that we can take a white noise process, filter it with f to give it the same colour as Y.

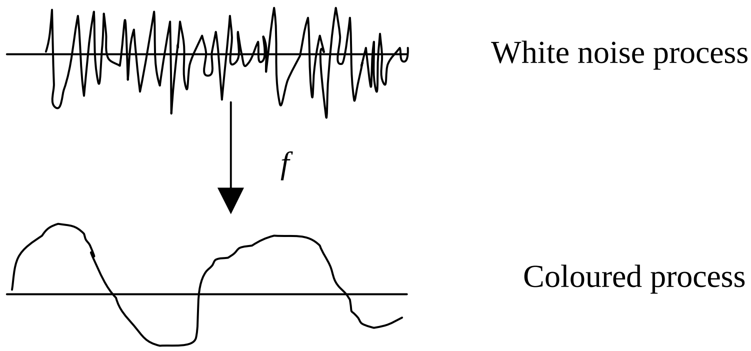

Figure1: A white noise process when filtered gives the coloured process

If f^{-1} exists, passing Y through f^{-1} we get an uncorrelated process Q, which will have the same information but has lesser energy. By having the same information, we mean that we can get Y back from the Q.

So how do we filter a stationary random process Y to get an uncorrelated process?

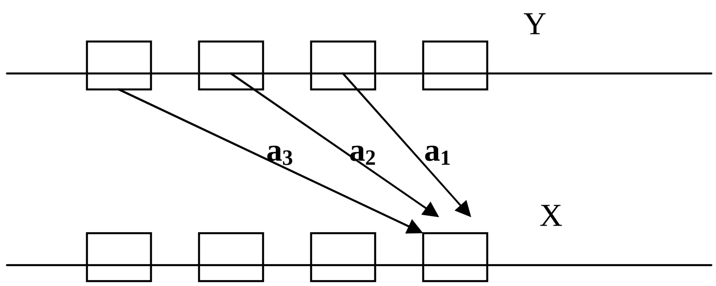

Figure2: Linear prediction of X using Y. $a_1, a_2, a_3, \ldots$ **are the filter coefficients.**

Well, if we do linear prediction, then the error we get is orthogonal to the signal used to estimate it. Lets use this fact and see what we can do:

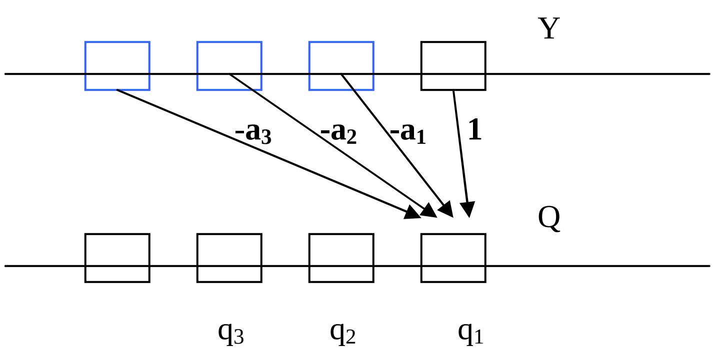

Figure3: The filter which gives the error corresponding to the optimal linear prediction

Looking at the above figure, we see that

1. q_1 is orthogonal to all the blue ys and thus to all vectors in the subspace of the blue ys..
2. q_2 is a linearly dependent on the same blue ys. And hence lie in the subspace of the blue ys.

From 1 and 2, we see that *q_1 is orthogonal to q_2* (and similarly to q_3, q_4, …).

Tadaa! We have Q, our uncorrelated process got by filtering Y. Q is often called the **innovations process** generating Y.

The filter $a_1, a_2, a_3,...$ is called the **Wiener filter**, and is the optimal linear prediction filter.
The filter $1, a_1, a_2, a_3,...$ is[3] the **Whitening filter**, and is the *prediction error filter*.
The inverse filter would be called the **Colouring filter**.

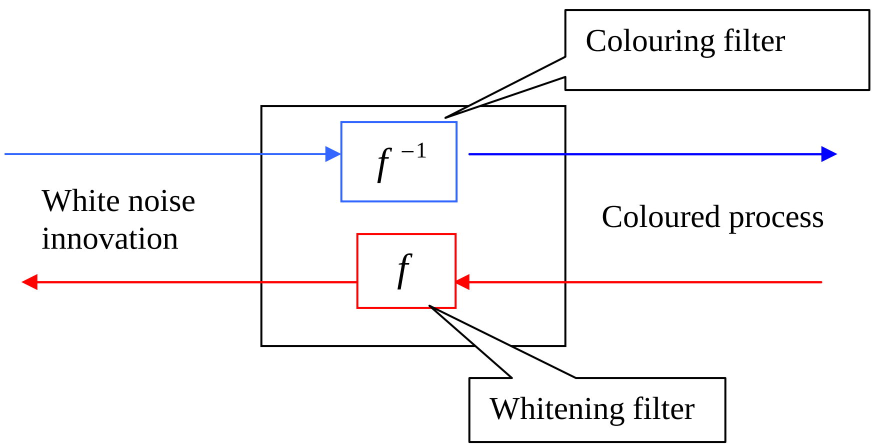

Figure4: Innovations representation of WSS process

Now what we have to show is that this Q exists for all stationary random processes i.e. the wiener filter exists, is causal and stable.

Before we go on to that, lets look at the **applications of the existence of such a whitening filter** f **.**
Suppose you have a speech signal to be transmitted. You pass it through a whitening filter to get the uncorrelated signal. Now you only send the filter coefficients across.
At the other end, you take an 'equivalent' white noise process, pass it thorigh f^{-1} to get the original speech signal. This may sound (no pun intended) a little fuzzy, but what they would do is model the innovations process of all speech signals in some way and use it; also the human ear is quite tolerant so the reconstructed speech signal would pass off as the same signal.

Anyway, getting back to the proof: fs
Given r_{ff} , we can find the filter f having that autocorrelation (where r_{ff} is γ_{YY}).

We are given[4]
$$f * \overleftarrow{f} = r_{ff}$$

Taking Fourier transform
$$F\overline{F} = R_{ff}$$

Great! We have the **magnitude**. If we put ***any*** **phase**, we're through!
Not quite.
We have shown that we can get the filter f , but *our main goal should be to get a causal stable filter whose inverse filter exists and is also causal and stable.*

Taking logarithms on both sides,
$$\log F + \log \overline{F} = \log R_{ff}$$

If,
$$q = re^{j\alpha}$$
$$\log q = \log r + j\alpha$$
$$\log \overline{q} = \log r - j\alpha$$
$$\therefore \log \overline{q} = \overline{\log q}$$

Using the above result,
$$\log F + \overline{\log F} = \log R_{ff}$$

Let
$$\log F = G$$
$$\log R_{ff} = S$$
$$G + \overline{G} = S$$

Taking Inverse Fourier transform
$$g + \overleftarrow{g} = s$$

We can see that s is symmetric
Let us take g to be the positive half of s.

What we will now show is that if g (called the ***cepstrum***) is causal, then so is f .
We will assume that G is analytic and is expressed by the Laurent series

$G = g_0 + g_1 z + g_2 z^2 + \ldots,$

which form the Fourier transform coefficients of g.

Now,

$F = \text{antilog}\, G$

This implies that F is also analytic and can be expressed as

$F = f_0 + f_1 z + f_2 z^2 + \ldots$

So $f_0, f_1, f_2, \ldots$ is the sequence that has F as its z-transform, which is the required **causal** filter f.

Further, G being analytic also implies that f is a *minimum phase filter*, which implies that if f is a rational function of z, then it has a **stable and causal inverse**.

Udayan's notes:

[1] "Shot Noise"

[2] "Inter symbol interference"!!!

[3] $1, -a_1, -a_2, -a_3, \ldots$

[4] The symbol $\overleftarrow{f}$ (read "f flipped") stands for the time-reversed f. $\overleftarrow{f}(n) = f(-n)$.