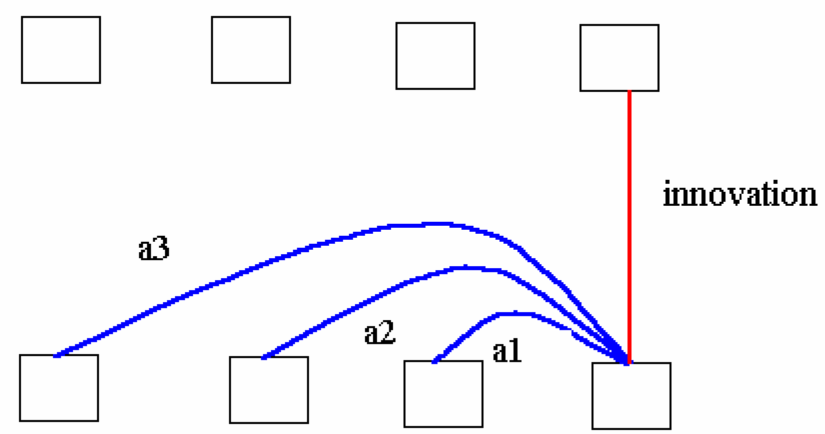
Weiner Filtering

WHAT IS AR RANDOM PROCESS?

AR process is a stationary random process whose coloring filter is Auto-Regressive(AR). Thus the present output is a linear function of previous outputs plus some innovation. The innovation can be white noise.



AR Random Process

AR(p) denotes a filter with ‘p’ poles for the corresponding AR random process. An AR(p) filter will have corresponding MA(p) inverse or whitening filter.

Now even though, we have infinitely long auto-correlation filter. But its corresponding de-correlating filter is finite.

POWER SPECTRUM ESTIMATION:

What is this?

Power spectrum estimation means estimating the Fourier Transform of auto-correlation function.

Why?

At a time, we’ll have only finite samples of a random process in hand and we’ll have to estimate the power spectrum from that may be for some prediction.

Ways of Power Spectrum Estimation:

1. Finding the auto-correlation function of the random process and taking its Fourier Transform.
2. Directly estimate the power spectrum.
3. First, we find the whitening filter. Then we invert it to get the corresponding coloring filter. From the coloring filter, we find the auto-correlation function of coloring filter which is the auto-correlation function of the stationary random process. In many applications, we don't have to go back to power spectrum from whitening filter. We can get Weiner filter directly from the whitening filter.
4. Estimate Reflection coefficients of lattice filter.

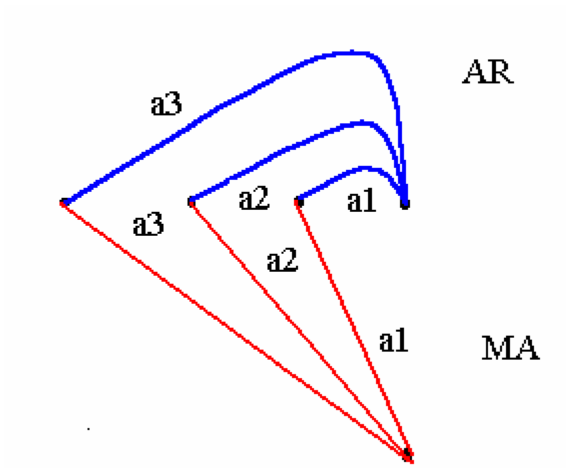
METHOD:

We have outcomes and we have an AR random process. We trying to estimate the coloring filter for this process provided order 'p' is known. We can do this using Least Square Deconvolution.

Prediction of new sample as linear combination of previous samples

$$\begin{pmatrix} x0 & x1 & x2 & x3 \\ x1 & x2 & x3 & x4 \\ x2 & x3 & x4 & x5 \\ x3 & x4 & x5 & x6 \end{pmatrix} \qquad =$$

We'll estimate using MA filter. But as input x and output x_{adv} are same. It is actually an AR filter.



AR coefficients = MA coefficients

Now we want best estimate with whatever samples we have in hand. Suppose, we have 25 samples viz $x_0 \ldots x_{24}$, then we'll get nice estimate for the 1-step correlation i.e. correlation between x_0 and x_1, x_1 and x_2 ….. and so on till x_{23} and x_{24}. This is because we have 24 pairs for the interval of 1. But for 23-step correlation value, we have just two pairs x_0 and x_{23}, x_1 and x_{24}. Thus more no of samples means better estimate.

So for LMS, we give x, x_{adv} and $x_{backward}$ to increase the no of samples. This is valid because stationary random processes look the same from backwards.

$$\begin{pmatrix} x0 & x1 & x2 & x3 \\ x1 & x2 & x3 & x4 \\ x2 & x3 & x4 & x5 \\ x3 & x4 & x5 & x6 \\ \hline x4 & x3 & x2 & x1 \\ x5 & x4 & x3 & x2 \end{pmatrix} \qquad = \qquad \underline{\quad}$$

Here a1,a2,a3,a4 are the coefficients of prediction filter. We can get whitening (prediction error filter) filter by putting 1 in present sample and negating previous coefficients.

Consider Z-domain,
Here the prediction filter will be-
$a_1z^{-1} + a_2z^{-2} + a_3z^{-3} + a_4z^{-4}$

the prediction error filter will be –

$1 - a_1z^{-1} - a_2z^{-2} - a_3z^{-3} - a_4z^{-4}$

the corresponding coloring filter will be –

$$\frac{1}{1 - a_1z^{-1} - a_2z^{-2} - a_3z^{-3} - a_4z^{-4}}$$

flipped coloring filter will be-

$$\frac{1}{1 - a_1z^{1} - a_2z^{2} - a_3z^{3} - a_4z^{4}}$$

We have to convolve these two filters to get the power spectrum in rational form. Convolution will be Multiplication in z-domain.

So the rational function for power spectrum will be-

$$\frac{1}{(1 - a_1z^{-1} - a_2z^{-2} - a_3z^{-3} - a_4z^{-4})(1 - a_1z^{1} - a_2z^{2} - a_3z^{3} - a_4z^{4})}$$

MA Processes:

For these processes, coloring filter is MA so corresponding whitening filter will be AR.

ARMA Processes:

For ARMA processes, both coloring and whitening filter will be ARMA with poles and zeros interchanged between them.

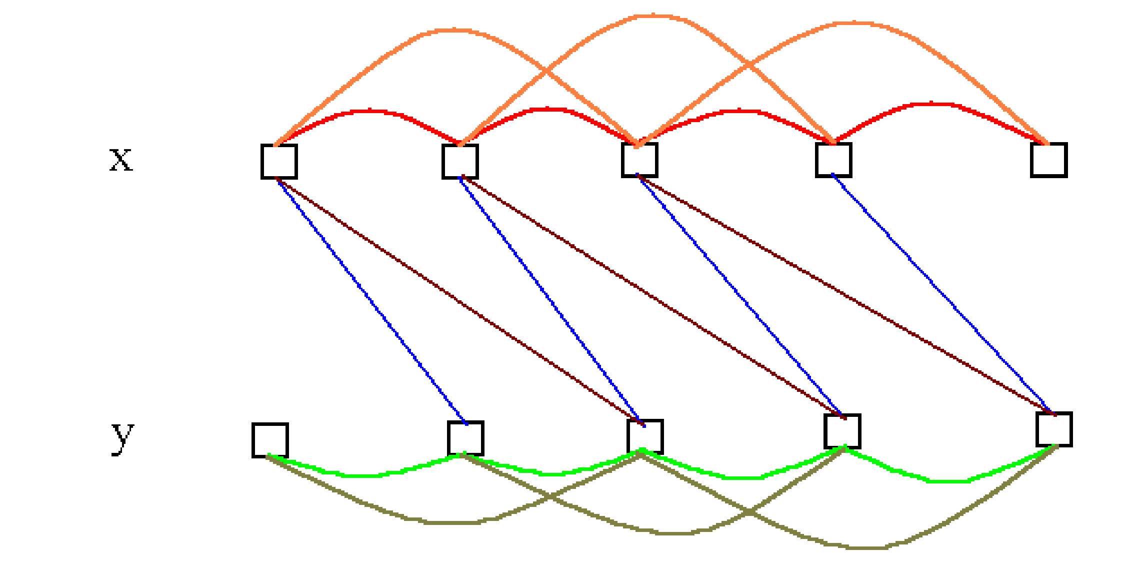
WEINER Filtering:

Weiner filtering is estimating a random process from few other random processes. Prediction is a special case of Weiner filtering where same random process is estimated in advance from previous samples.

Jointly Stationary Random Processes:

X and Y are said to be Jointly Stationary Random Processes if and only if-

1. If X is a stationary random process.
2. If Y is a stationary random process.
3. If the cross-correlation $r_{xy}(k,l)$ depends only on the difference (k-l).



Jointly Stationary Random Processes X and Y

In the above figure, we can see two jointly stationary random processes X and Y. The correlation values shown by same color are equal. So we can see that X and Y are independently stationary plus their cross-correlation is also same for same interval. This makes them Jointly stationary random processes.

If two processes are jointly stationary, then we can design the Weiner filter for them.

FIR Weiner Filter:

If we have two jointly stationary random processes X and Y and we want to predict Y from the samples of X. We now have auto-correlation of X and cross-correlation between X and Y.

Here

The auto-correlation of X is given by-

$$E\{x(n-l)x^*(n-k)\} = r_x(k-l)$$

The cross-correlation of X and Y is given by-

$$E\{y(n)x^*(n-k)\} = r_{xy}(k)$$

Now if ‘w’ is the Weiner filter response, we have-

$$w * r_{xx} = r_{xy}$$

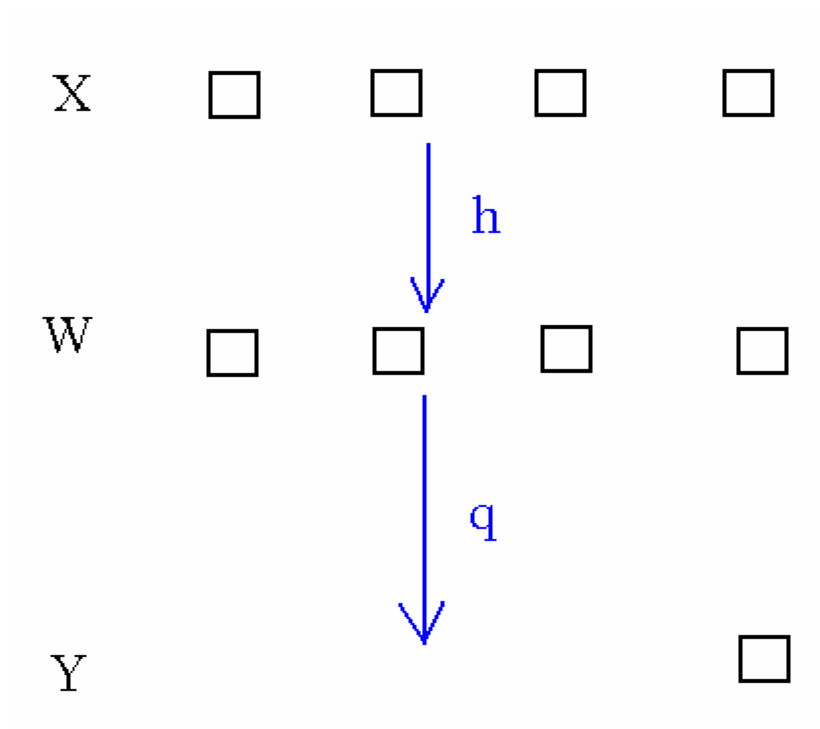
$$\begin{pmatrix} rx(0) & rx^*(1) & \ldots\ldots & rx^*(p-1) \\ rx(1) & rx(0) & \ldots\ldots & rx^*(p-2) \\ rx(2) & rx(2) & \ldots\ldots & rx^*(p-3) \\ . & & \ldots\ldots & \\ . & & \ldots\ldots & \\ rx(p-1) & rx(p-2) & \ldots\ldots & rx(0) \end{pmatrix} \begin{pmatrix} w(0) \\ w(1) \\ w(2) \\ . \\ . \\ w(p-1) \end{pmatrix} = \begin{pmatrix} rxy(0) \\ rxy(1) \\ rxy(2) \\ . \\ . \\ rxy(p-1) \end{pmatrix}$$

so

$$W = E[XX^T]^{-1} E[XY]$$

Causal IIR Weiner Filter:

Now we want to design a causal IIR Weiner filter for prediction of a random process Y from samples of another random process X.



We first calculate W which is filtered version of X. W is the innovation process of X. Now X and Y are jointly stationary, hence W and Y are also jointly stationary.

Now we have cross-correlation of X and Y. But we need cross-correlation of W and Y.

$E[w_{k-i}\, y_n] \quad = E[\,(x_{n-i}h_0 + x_{n-i-1}h_1 + x_{n-i-2}h_2 + \ldots)\, y_n]$

$= h_0 E[x_{n-i} y_n] + h1 E[x_{n-i-1} y_n] + h E[x_{n-i-2} y_n] + \ldots$

$r_{wy}(i) = h_0 r_{xy}(-i) + h_1 r_{xy}(-i-1) + \ldots\ldots$

$r_{wy} = h * r_{xy}$

$q(i) = E[W(n-i)y(n)] = r_{wy}(-i)$

as we want causal filter only,

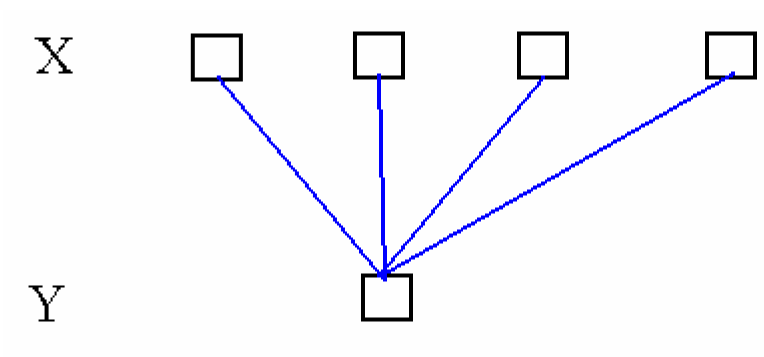
q = [flipped r_{wy}]+

= [(flipped r_{xy}) * (flipped h)]+

h*q = h * [(flipped r_{xy}) * (flipped h)]+

Here h and q both are causal so even (h * q) is also causal.

UNREALIZABLE Weiner filter (Non-Causal):



Non-Causal Weiner Filter

Here we have all boxes(analogy!) open. So we know all samples of X. So now comapared to previous case of causal filter, we have

q = [flipped r_{wy}]

h*q = h * [(flipped r_{xy}) * (flipped h)]

h*q = h * (flipped h)* (flipped r_{xy})

So in Z-domain –

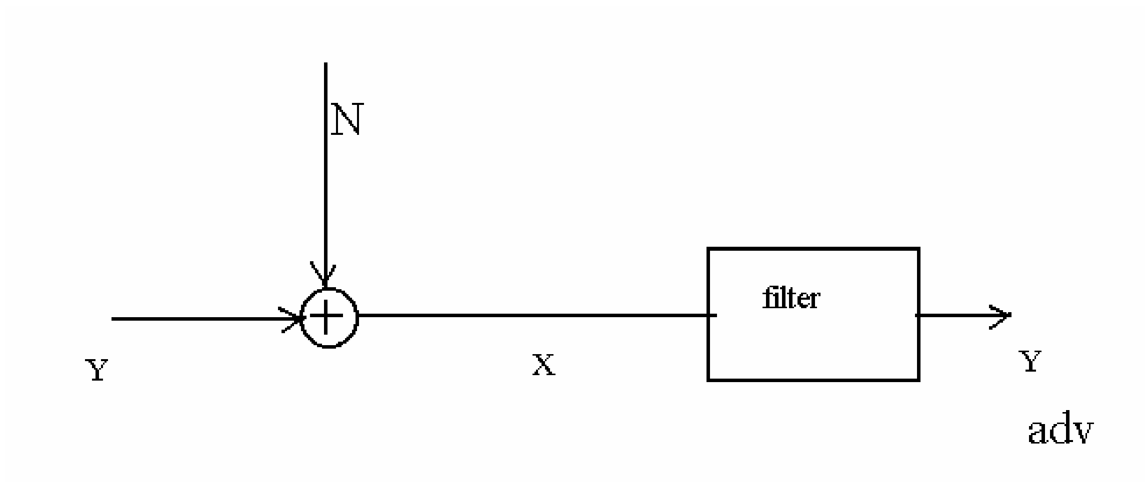
$H(z)Q(z) = R_{hh}\Gamma yx$

let 'g' be inverse filter of h. Then R_{gg} will be auto-correlation of X.

$= \Gamma yx / R_{gg}$

$$\mathbf{H(z)Q(z)=\Gamma yx / \Gamma xx}$$

Weiner Filtering for noise removal:



Removal of noise from the past samples is also called smoothing.

$E[X_t X_{t-i}] = E[(y_t + N_t)(y_{t-i} + N_{t-i})]$

$= E[(y_t\ y_{t-i})] + E[(N_t\ N_{t-i})] + E[(y_t\ N_{t-i})] + E[(y_{t-i}\ N_t)]$

$r_{xx}(i) = r_{yy}(i) + r_{NN}(i) + r_{yN}(i) + r_{Ny}(i)$

Now last two terms are zero cause noise is most of the times un-correlated with signal Y.

If noise is white noise, then $r_{NN}(i)$ is a delta function.

$E[Y_t X_{t-i}] = r_{xy}(i)$

$= E[Y_t Y_{t-i}] + E[Y_t N_{t-i}]$

$r_{xy}(i) = r_{yy}(i) + r_{yN}(i)$